

Program Specialisation and Abstract Interpretation Reconciled

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Report CW 259, May 1998



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Abstract

We clarify the relationship between abstract interpretation and program specialisation in the context of logic programming. We present a generic top-down abstract specialisation framework, along with a generic correctness result, into which a lot of the existing specialisation techniques can be cast. The framework also shows how these techniques can be further improved by moving to more refined abstract domains. It, however, also highlights inherent limitations shared by all these approaches. In order to overcome them, and to fully unify program specialisation with abstract interpretation, we also develop a generic combined bottom-up/top-down framework, which allows specialisation *and* analysis outside the reach of existing techniques.

Keywords : Program Specialisation, Abstract Interpretation, Partial Deduction, Partial Evaluation, Program Transformation, Logic Programming.

CR Subject Classification : I.2.2, D.1.2, I.2.3, F.4.1, D.1.6.

Program Specialisation and Abstract Interpretation Reconciled

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Abstract

We clarify the relationship between abstract interpretation and program specialisation in the context of logic programming. We present a generic top-down abstract specialisation framework, along with a generic correctness result, into which a lot of the existing specialisation techniques can be cast. The framework also shows how these techniques can be further improved by moving to more refined abstract domains. It, however, also highlights inherent limitations shared by all these approaches. In order to overcome them, and to fully unify program specialisation with abstract interpretation, we also develop a generic combined bottom-up/top-down framework, which allows specialisation *and* analysis outside the reach of existing techniques. This technical report is an extended version of [24].

1 Introduction

At first sight *abstract interpretation* (see, e.g., [6, 3]) and *program specialisation* (see, e.g., [11]) might appear to be completely unrelated techniques: abstract interpretation focusses on *correct and precise* analysis, while the main goal of program specialisation is to produce more *efficient residual code* (for a given task at hand). Nonetheless, it is often felt that there is a close relationship between abstract interpretation and program specialisation and, recently, there has been a lot of interest in the integration of these two techniques (see, e.g., [5, 26, 18, 36]).

Indeed, for good specialisation to take place, program specialisers have to perform some form of analysis. For instance, the incomplete SLD-trees produced by *partial deduction* [31, 11, 20] can be seen as complete (given the closedness condition of [31]) description of the top-down computation-flow.

In this paper we want to substantiate this intuition and make the link to abstract interpretation fully explicit. We therefore present a generic (augmented) top-down abstract interpretation framework in which most of the specialisation techniques (such as partial deduction [31, 11, 20], ecological partial deduction [21, 28, 22], constrained partial deduction [25], conjunctive

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partial deduction [27, 15]) can be cast. It also paves the way for more refined and powerful specialisation, by allowing more refined abstract domains and more refined “abstract unfolding” rules.

However, we will not stop there. The above formalisation will actually make two (additional) shortcomings of most earlier specialisation techniques apparent (already identified in, e.g., [26, 22]): the lack of *side-ways information passing* and of *inference of global success information*. Recent techniques [26, 37, 36] (as well as some earlier attempts such as [32] and [14, 7]) have tried to overcome these limitations by incorporating *bottom-up* abstract interpretation techniques. See also [42] which achieves a form of bottom-up information propagation in another manner. However, we feel that a fully satisfactory integration of program specialisation with abstract interpretation has not been achieved yet, and we strive to do so in this paper.

This integration is not solely beneficial for specialisation purposes. Indeed, as shown in [26, 22] (for one particular abstract domain), a full integration of abstract interpretation with program specialisation can yield *analysis* outside the reach of either method alone (and can even be used to perform inductive theorem proving or infinite model checking[23]).

We thus present an augmented, combined top-down/bottom-up abstract specialisation framework in which all these earlier techniques (and more) can be cast, and provide the first “full-blown” integration of abstract interpretation and program specialisation, leading towards powerful specialisation *and* analysis beyond the reach of existing techniques.

2 Top-Down Abstract Partial Deduction

In this paper, we restrict ourselves to definite programs and goals (but possibly with declarative built-in’s such as *is*, *call*, *functor*, *arg*, $\backslash ==$; this allows to express a very large number of interesting, practical programs; one can even implement and use certain higher-order features such as *map/3*).

An *expression* is either a term, an atom or a conjunction. We use $E_1 \preceq E_2$ to denote that the expression E_1 is an instance of the expression E_2 . By $mgu(E_1, E_2)$ we denote a most general unifier and by $msg(E_1, E_2)$ a most specific generalisation of E_1 and E_2 . Also, as common in partial deduction, the notion of SLD-trees is extended to allow *incomplete* SLD-trees which may contain leaves where no literal has been selected for further derivation.

2.1 The Abstract Domain

We denote by \mathcal{Q} the set of all conjunctions. Our abstract domain \mathcal{AQ} is then a set of *abstract conjunctions* equipped with a (total)² *concretisation function* $\gamma : \mathcal{AQ} \mapsto 2^{\mathcal{Q}}$, providing the link between the abstract and the concrete domain. We suppose that $\gamma(\mathbf{A})$ is always *downwards closed*

²Unless explicitly stated otherwise, all functions will be considered total.

$(Q \in \gamma(\mathbf{A}) \Rightarrow Q\theta \in \gamma(\mathbf{A}))$, i.e. we restrict ourselves to “declarative” properties. This will be vital in our correctness proofs.³ Also, for reasons that will become clear below, we suppose that all conjunctions in $\gamma(\mathbf{A})$ have the same number of conjuncts and with the same predicates at the same position. Observe that this still admits the possibility of a bottom element \perp whose concretisation is empty. We will also need the following auxiliary concepts.

We will denote the fact that $\gamma(\mathbf{A}_1) \subseteq \gamma(\mathbf{A}_2)$ by $\mathbf{A}_1 \sqsubseteq \mathbf{A}_2$. In abstract interpretation one often requires $\mathcal{A}\mathcal{Q}, \sqsubseteq$ to form a lattice. For our purposes, this aspect is not essential. We also sometimes use γ on sets of abstract conjunctions: $\gamma(S) = \{Q \mid Q \in \gamma(\mathbf{A}) \wedge \mathbf{A} \in S\}$. One particular abstract domain, which will often serve for illustration purposes, is the \mathcal{PD} -domain where $\mathcal{A}\mathcal{Q} = \mathcal{Q}$ (i.e. the abstract conjunctions are the concrete ones) and $\gamma(Q) = \{Q' \mid Q' \preceq Q\}$ (i.e. an abstract conjunction denotes all its instances).

2.2 Abstract Unfolding

Program specialisation can achieve more efficient residual code — amongst others — by pre-computing certain operations at compile time (which then no longer have to be performed at run-time). In other words, one computation step in the residual program may actually represent an entire sequence of computation steps within the original program.

In the context of logic programming, this can be seen as producing a *residual* clause which, when resolved against, has the same effect as a sequence of resolution steps in the original program. Partial deduction, for example, produces these clauses by *unfolding* an atom A , thereby producing an SLD-tree τ for $P \cup \{\leftarrow A\}$. Every non-failing branch of τ is translated into a residual clause by taking the *resultant* of the derivation.⁴ These resultants can then be used in a sound manner for any concretisation of A (i.e., any instance of A) in the sense that resolution will lead to computed answers and resolvents which can also be obtained in the original program. But actually the use by partial deduction algorithms of these resultants is not limited to code generation. Take, e.g., the resultant $p(f(X)) \leftarrow q(f(X))$. When resolving a particular runtime call $p(\bar{t})$ with that resultant we will obtain resolvents which are instances of $\leftarrow q(f(X))$. Partial deduction therefore also analyses (and specialises) the atom $q(f(X))$. In other words, the body of the residual clause is used for the flow analysis as a representative of all possible resolvents. This multiple use of the residual clauses relies on using an abstract domain identical to the concrete domain.

In the more general setting we endeavour to develop, these two roles of unfolding will have to be separated out (as the residual program has to be expressed in the concrete domain). In other words, to specialise an abstract

³Anyway, in a purely declarative setting, it is difficult to imagine how one could exploit non-downwards closed properties for the code generation.

⁴A resultant is a formula $H \leftarrow B$ where H and B are conjunctions of literals. The resultant of a derivation of $P \cup \{\leftarrow Q\}$ with c.a.s. θ leading to B is the formula $Q\theta \leftarrow B$.

conjunction \mathbf{A} we generate:

- resultants $H_i \leftarrow B_i$, totally correct for the calls in $\gamma(\mathbf{A})$ (**abstract unfolding**) and
- for each resultant $H_i \leftarrow B_i$ an abstract conjunction \mathbf{A}_i approximating all the possible resolvent goals which can occur after resolving an element of $\gamma(\mathbf{A})$ with $H_i \leftarrow B_i$ (**abstract resolution**).

This leads to Definition 2.1 of abstract unfolding and resolution below. (Observe that the resultants $H_i \leftarrow B_i$ below are not necessarily Horn clauses. We will discuss the generation of Horn clause programs later in Section 2.4.) First, we introduce the following notations. Given an SLD-tree τ for $P \cup \{\leftarrow Q\}$ we denote by $Q \rightsquigarrow_{\tau}^{\theta} L$ the fact that a leaf goal L of τ can be reached via c.a.s. θ . Given a resultant $C_i = H_i \leftarrow B_i$ and a conjunction Q we denote by $Q \rightsquigarrow_{C_i}^{\theta} L$ the fact that $mgu(Q, H_i) = \theta'$ with $\theta' \upharpoonright_{vars(Q)} = \theta$ and $L = B_i\theta'$.⁵

Definition 2.1 An *abstract unfolding* operation $aunf(\cdot)$ maps abstract conjunctions to finite sets of resultants and has the property that for all $\mathbf{A} \in \mathcal{AQ}$ and $Q \in \gamma(\mathbf{A})$ there exists an SLD-tree τ for $P \cup \{\leftarrow Q\}$ such that:

$$Q \rightsquigarrow_{\tau}^{\theta} L \Leftrightarrow \exists C_i \in aunf(\mathbf{A}) \text{ s.t. } Q \rightsquigarrow_{C_i}^{\theta} L \quad (1)$$

An *abstract resolution* operation $ares(\cdot)$ maps an abstract conjunction \mathbf{A} and a concrete resultant C_i to another abstract conjunction such that for all $Q \in \gamma(\mathbf{A})$:

$$Q \rightsquigarrow_{C_i}^{\theta} L \Rightarrow L \in \gamma(ares(\mathbf{A}, C_i)) \quad (2)$$

The \Rightarrow part of point 1 requests that the code generated by $aunf(\cdot)$ is *complete* while the \Leftarrow part additionally requests *soundness* (as we want to have residual code which is *totally correct* and not just a safe approximation). We call an abstract unfolding rule *conservative* if the \Leftarrow part of point 1 holds for all Q (and not just for $Q \in \gamma(\mathbf{A})$).

The following examples illustrate several ways to perform abstract unfolding.

Example 2.2 Let P be the following program:

$$\begin{aligned} &eq([], []) \leftarrow \\ &eq([H|X], [H|Y]) \leftarrow eq(X, Y) \end{aligned}$$

Let $\mathbf{A} = eq([a|T], Z)$ in the \mathcal{PD} -domain. $aunf(\mathbf{A}) = \{H \leftarrow B\}$ and $ares(\mathbf{A}, H \leftarrow B) = eq(T, Y)$ where $H = eq(X, [a|Y])$ and $B = eq(X, Y)$ are correct. Also, both remain correct with $H = eq(H, [H|Y])$ but not with $H = eq(X, [b|Y])$. $H = eq([H], [H|Y])$ and $B = eq([], Y)$ is also incorrect.

⁵If Q and H_i are atoms this is equivalent to saying that $\leftarrow Q$ resolves with the clause $H_i \leftarrow B_i$ via c.a.s. θ yielding $\leftarrow L$ as resolvent. Also observe that for any Q, C_i and there is at most one choice of θ and L such that $Q \rightsquigarrow_{C_i}^{\theta} L$ (i.e., we suppose that the unification is fixed; this is important for the correctness criterion of abstract unfolding).

Observe, that in Definition 2.1 above, nothing forces one to use the *same* structure (i.e. same selected literal positions, same clauses) for *all* the concretisations of \mathbf{A} . Indeed, this enables some very powerful optimisations not achievable within existing “classical” specialisation frameworks. For instance, in the example below we are able to completely eliminate a type-like check from the residual program.

Example 2.3 Let P be the program from Example 2.2 and \mathbf{A} represent the set of all calls $eq(L, L)$ where L is a bounded (nil-terminated) list (this can obviously not be represented in the \mathcal{PD} -domain). Then $aunf(\mathbf{A}) = C_1 = \{eq(X, Y) \leftarrow\}$ and $ares(\mathbf{A}, C_1) = \square$ are correct according to the above definition ! One can thus generate the residual code:

$$eq(X, Y) \leftarrow$$

Observe that this abstract unfolding is, in contrast to Example 2.2, not conservative. In other words the residual code is only sound for concretisations of \mathbf{A} but not, e.g., for the call $eq(a, [])$.

Example 2.4 Let P be the following program:

$$\begin{aligned} (C_1) \quad & p(a) \leftarrow \\ (C_2) \quad & p(f(X)) \leftarrow p(X) \\ (C_3) \quad & p(g(X)) \leftarrow p(X) \end{aligned}$$

Let \mathbf{A} represent all calls $p(X)$ where X has type $\tau ::= a \mid g(\tau)$. Then $aunf(\mathbf{A}) = \{C_1, C_3\}$, $ares(\mathbf{A}, C_1) = \square$ and $ares(\mathbf{A}, C_3) = \mathbf{A}$ is correct and by abstract unfolding we are able to safely remove the redundant clause C_2 .

To more concisely express the flow analysis, we extend $aunf(\cdot)$ so that it maps sets of abstract conjunctions to sets of abstract conjunctions in the following way: $aunf^*(S) = \{ares(\mathbf{A}, C) \mid C \in aunf(\mathbf{A})\}$. (This is actually the operation that would be sufficient to perform “ordinary” top-down abstract interpretation without specialisation.)

2.3 Widening by Splitting

The computation flow aspect of program specialisation could now be performed by calculating $U \uparrow^\infty$, where $U(S) = S \cup aunf^*(S)$. However, it is obvious that, for but the simplest abstract domains, this construction will not terminate and that generalisation is required.

As usual in abstract interpretation, one could imagine to represent generalisation by a widening function $\omega : \mathcal{AQ} \mapsto \mathcal{AQ}$ such that $\forall \mathbf{A} \in \mathcal{AQ} : \mathbf{A} \sqsubseteq \omega(\mathbf{A})$. Unfortunately, this is not enough to be able to ensure termination of abstract interpretation in the present setting, because all concretisations of an abstract conjunction must have the same number of conjuncts. In other words, no terminating analysis could be produced for, e.g., a program containing the clause $p \leftarrow p, p$. This is why we need a more refined notion of widening, which involves splitting conjunctions into subconjunctions:

Definition 2.5 A sequence $\langle \mathbf{A}_1, \dots, \mathbf{A}_n \rangle$ of abstract conjunctions is an *abstraction* of an abstract conjunction \mathbf{A} iff $\gamma(\mathbf{A}) \subseteq \{Q_1 \wedge \dots \wedge Q_n \mid Q_i \in \gamma(\mathbf{A}_i)\}$.

Observe that for $i = 1$ this condition is equivalent to $\mathbf{A} \sqsubseteq \mathbf{A}_1$. Also observe that this splitting operation does not allow re-ordering of conjunctions.⁶ It is, however, straightforward to do so. One just has to be careful to use the *same* reordering for *all* concretisations (otherwise it will be impossible to synchronise the code generation with the flow analysis, cf. the next subsection).

We extend the abstraction concept to sets:

Definition 2.6 A set \mathcal{A}' is called an *abstraction* of another set of abstract conjunctions \mathcal{A} , denoted by $\mathcal{A}' \sqsupseteq_{split} \mathcal{A}$, iff for all $\mathbf{A} \in \mathcal{A}$ there exists an abstraction $\langle \mathbf{A}_1, \dots, \mathbf{A}_n \rangle$ of \mathbf{A} such that all $\mathbf{A}_i \in \mathcal{A}'$.

For example, in the \mathcal{PD} -domain, $\langle p(X) \wedge q(X), p(b) \rangle$ is an abstraction of $p(b) \wedge q(b) \wedge p(b)$ and we have thus, for example, $\{p(X) \wedge q(X), p(b)\} \sqsupseteq_{split} \{p(b) \wedge q(b) \wedge p(b), p(c) \wedge q(c)\}$.

We can now define a more refined *widening operator* to be a function⁷ $\omega : 2^{\mathcal{A}\mathcal{Q}} \mapsto 2^{\mathcal{A}\mathcal{Q}}$ satisfying that $\omega(\mathcal{A}) \sqsupseteq_{split} \mathcal{A}$ for all \mathcal{A} .

By using appropriate widening operators it is now possible to ensure termination for any program (we refer the reader, e.g., to [28, 15, 22] on how to devise ω in the context of partial deduction).

We also say that a set \mathcal{A} of abstract conjunctions is *covered* iff $\mathcal{A} \sqsupseteq_{split} aunf^*(\mathcal{A})$. Intuitively, this means that \mathcal{A} is a complete description of the computation flow (induced by $aunf(\cdot)$) for all concretisations of \mathcal{A} .

2.4 Generating Residual Code

Generating residual code from the resultants $H_i \leftarrow B_i$ produced by the abstract unfolding involves transforming them into Horn clauses. This can be achieved by mapping the abstract conjunctions produced by the flow analysis to atoms and then appropriately renaming the heads H_i and the bodies B_i .

We first introduce the following concepts.

Definition 2.7 A *concrete dominator* of an abstract conjunction \mathbf{A} is a concrete conjunction which is more general than all the concretisations of \mathbf{A} . A *skeleton* for an abstract conjunction \mathbf{A} is a maximally general concrete dominator of \mathbf{A} .

⁶Nor removal of duplicate calls. In general this does not preserve computed answers (but will produce more general answers) but is, e.g., required for tupling the Fibonacci function. It is quite straightforward to add this possibility to the framework.

⁷It is of course possible to give extra parameters to ω , e.g., to take the specialisation history into account.

By our earlier assumption that all conjunctions in $\gamma(\mathbf{A})$ have the same predicates at the same position we know that a concrete dominator (and thus skeleton) exists for all abstract conjunctions. By $\lceil \mathbf{A} \rceil$ we denote some skeleton for \mathbf{A} .

We also require that for all $\mathbf{A} \in \mathcal{AQ}$ and $H_i \leftarrow B_i \in \text{aunf}(\mathbf{A})$ we have $H_i \preceq \lceil \mathbf{A} \rceil$. The requirement prevents garbage code (any $H_i \not\preceq \lceil \mathbf{A} \rceil$ can never unify with a concretisation of \mathbf{A}) and simplifies the construction below.

Definition 2.8 An *atomic renaming* $\rho_{\mathbf{A}}$ for an abstract conjunction \mathbf{A} is an atom A such that $\text{vars}(\lceil \mathbf{A} \rceil) = \text{vars}(A)$. Also, for any $Q \preceq \lceil \mathbf{A} \rceil$ we define $\rho_{\mathbf{A}}(Q) = A\theta$ where θ is such that $Q = \lceil \mathbf{A} \rceil\theta$.

In the \mathcal{PD} -domain, we might have $\mathbf{A} = p(f(X)) \wedge q(Z)$, $\lceil \mathbf{A} \rceil = p(X) \wedge q(Y)$, $\rho_{\mathbf{A}} = pq(X, Y)$ and $Q = p(f(a)) \wedge q(b)$. In that case $\rho_{\mathbf{A}}(Q) = pq(f(a), b)$.

Observe that for all $Q \preceq \lceil \mathbf{A} \rceil$ we have $\rho_{\mathbf{A}}(Q\theta) = \rho_{\mathbf{A}}(Q)\theta$ and for all $Q' \preceq \lceil \mathbf{A} \rceil$ we also have $\text{mgu}(Q, Q') = \text{mgu}(\rho_{\mathbf{A}}(Q), \rho_{\mathbf{A}}(Q'))$. Also, to avoid name clashes, we will always suppose that for any $\mathbf{A} \neq \mathbf{A}'$ the predicate symbols used by $\rho_{\mathbf{A}}$ and $\rho'_{\mathbf{A}'}$ are different.

Given a resultant $H_i \leftarrow B_i \in \text{aunf}(\mathbf{A})$ we can now produce an actual Horn clause by renaming H_i and B_i . Renaming H_i is easy: we just calculate $\rho_{\mathbf{A}}(H_i)$ (which is always defined). If our flow analysis also contains $\mathbf{A}_i = \text{ares}(\mathbf{A}, H_i \leftarrow B_i)$ (and thus code for \mathbf{A}_i will be generated) then renaming B_i is just as easy: we just calculate $\rho_{\mathbf{A}_i}(B_i)$. However, suppose that we have applied a widening step and that we actually did not analyse \mathbf{A}_i but an abstraction $\langle \mathbf{G}_1, \dots, \mathbf{G}_n \rangle$ of it. In that case B_i has to be chopped up and then renamed using the renaming functions of the abstraction. We thus define $\rho_{\mathcal{A}, \mathbf{A}}(B) = \rho_{\mathbf{G}_1}(B_1) \wedge \dots \wedge \rho_{\mathbf{G}_n}(B_n)$ where $B = B_1 \wedge \dots \wedge B_n$ and $\langle \mathbf{G}_1, \dots, \mathbf{G}_n \rangle$ is an abstraction of \mathbf{A} such that $\mathbf{G}_i \in \mathcal{A}$ and $B_i \preceq \lceil \mathbf{G}_i \rceil$. If no such partitioning exists then we leave $\rho_{\mathcal{A}, \mathbf{A}}(B)$ undefined.

Definition 2.9 Let \mathcal{A} be a covered set of abstract conjunctions. We then define an *abstract partial deduction of P wrt \mathcal{A}* to be the set of clauses:

$$\{\rho_{\mathbf{A}}(H) \leftarrow \rho_{\mathcal{A}, \mathbf{A}'}(B) \mid H \leftarrow B \in \text{aunf}(\mathbf{A}) \wedge \mathbf{A}' = \text{ares}(\mathbf{A}, H \leftarrow B) \wedge \mathbf{A} \in \mathcal{A}\}.$$

It is easy to see that, because \mathcal{A} be a covered, the renamings of the bodies B will always be defined.

Observe that, a skeleton always has distinct variables as its only terms. In other words, we perform no structure filtering (i.e. $p(f(a))$ might get renamed into $p'(f(a))$ but never into $p'(a)$ or p'). Filtering could be achieved by using a concrete dominator, ideally $\text{msg}(\gamma(\mathbf{A}))$, instead of the skeleton $\lceil \mathbf{A} \rceil$ for the definition of $\rho_{\mathbf{A}}$. This, however, makes the exposition more tricky⁸ and would detract from the main points of the paper. Anyway, one can always apply [12] (as well as [29]) as a post-processing.

⁸Indeed, although all concretisations of \mathbf{A} will be an instance of $\text{msg}(\gamma(\mathbf{A}))$, this does not necessarily hold for the heads H and bodies B generated by the abstract unfolding.

2.5 A Generic Correctness Result and Algorithm

We can now present a very general correctness result.

Theorem 2.10 Let P' be an abstract partial deduction of P wrt a covered set of abstract conjunctions \mathcal{A} and let $Q \in \gamma(\mathbf{A})$ with $\mathbf{A} \in \mathcal{A}$. Then $P \cup \{\leftarrow Q\}$ has an SLD-refutation with c.a.s. θ iff $P' \cup \{\leftarrow \rho_{\mathbf{A}}(Q)\}$ has an SLD-refutation with c.a.s. θ .

Proof In Appendix A. □

In order to derive results about the preservation of finite failure we have to impose that the unfolding operation $aunf(\cdot)$ is *fair*⁹, i.e. when computing $aunf(\mathbf{A})$ it eventually selects every conjunct of $Q \in \gamma(\mathbf{A})$ in every non-failing branch. One can then prove (by reusing results from [27, 22]) that:

$P \cup \{\leftarrow Q\}$ has a finitely failed SLD-tree iff $P' \cup \{\leftarrow \rho_{\mathbf{A}}(Q)\}$ has.

Based upon the notions introduced above, we can now present a generic algorithm for top-down program specialisation in a very concise manner:

Algorithm 2.11 (Top-Down Abstract Partial Deduction)

Input: A program P and an abstract conjunction \mathbf{A}

Output: A specialised program P'

Initialise: $i = 0$, $\mathcal{A}_0 = \{\mathbf{A}\}$

repeat

let $\mathcal{A}_{i+1} := \omega(\mathcal{A}_i \cup aunf^*(\mathcal{A}_i));$ **let** $i := i + 1;$

until $\mathcal{A}_{i-1} = \mathcal{A}_i$

 Let P' be an abstract partial deduction wrt \mathcal{A}_i

The differences over “traditional” top-down abstract interpretation methods for logic programs (like, e.g., the top-down component of [3]) are:

1. abstract *conjunctions* instead of abstract atoms are used,
2. widening can generalise by “going up the lattice” *and* by splitting,
3. a *full abstract unfolding* (which can do, e.g., deforestation) is used instead of just a single abstract resolution step, and
4. there is no sideways information passing between abstract conjunctions (but perfect [26] sideways propagation within each abstract conjunction).

Observe that, in practical algorithms, ω might actually maintain some structuring of the elements in \mathcal{A}_i (e.g. the global trees used in [28, 15]) and therefore actually take some hidden arguments.

⁹Or even better weakly fair, see [27, 22].

2.6 Expressing Existing Partial Deduction Techniques

Classical partial deduction [31, 11] can be seen as an instance of the above generic framework by taking

- the \mathcal{PD} -domain (i.e. the concrete domain is the abstract domain and an abstract element represents all its instances),
- abstract unfolding performs concrete resolution steps,
- ω will only produce sets of atoms and the initial abstract conjunction \mathbf{A} is an atom.

To represent *conjunctive* partial deduction [27, 15, 22] we just have to drop the last requirement. *Ecological* partial deduction [21, 28, 22] can be seen as an instance of the above generic framework by taking

- $\mathcal{AQ} = (\mathcal{A}, T)$, where \mathcal{A} is the set of atoms and T is the set of characteristic trees [13, 10].
- $\gamma((A, \tau)) = \{A'' \mid A'' \preceq A' \preceq A \wedge A' \text{ has characteristic tree } \tau\}$,
- $\text{unf}((A, \tau))$ is based on using the SLD-tree τ (see [21, 28, 22]).

Similarly, *constrained* partial deduction [25] can be cast into the present framework (e.g. $\gamma(c \square A) = \{A\theta \mid \mathcal{D} \models \forall(c\theta)\}$), and its correctness results are a special case of the ones above.

The present framework can now be used to easily extend both methods to handle conjunctions or even to integrate all of these methods into one powerful top-down specialisation method.

2.7 Future Prospects

Improved Generalisation A lot of existing specialisation techniques (see, e.g., [39, 28, 15]) ensure termination by using refined methods, such as homeomorphic embedding \trianglelefteq , to detect “dangerous” growth of structure. However, once such a growth has been detected these techniques still have to rely on rather crude generalisation operators, such as more specific generalisation (*msg*), because the resulting generalisation (or part of it, as in [28]) has to be expressed in the \mathcal{PD} -domain. For instance, when a specialiser goes from $A_1 = p(a)$ to $A_2 = p(f(a))$ then the homeomorphic embedding \trianglelefteq will signal danger ($A_1 \trianglelefteq A_2$) and will even pinpoint the extra $f(\cdot)$ in A_2 as the potential source of non-termination. But the *msg* of A_1 and A_2 is just $p(X)$ and no use of the information provided by \trianglelefteq was made (nor is it possible to do so in the \mathcal{PD} -domain). In the enriched context of abstract partial deduction, however, we can now derive, e.g., a (regular) *type* describing the growth detected by \trianglelefteq and arrive at much more intelligent generalisation and much improved specialisation. For instance A_1 and A_2 can be abstracted by something like $p(X:\tau)$ where the type τ is defined as $\tau ::= a \mid f(\tau)$. Also, atoms such as $p([\])$ and $p([H|T])$ can be abstracted by $p(X:\text{list})$.¹⁰ The example worked out in Section 4 will make use of that possibility.

¹⁰In order to develop a practical algorithm for such an abstract domain, one would of course also have to be able to generalise types (e.g. based on calculating the union type).

Improved Unfolding and Code Generation As already hinted at in Section 2.2, our enriched abstract unfolding operation allows us to generate much more efficient code. Given the simple program

$$\begin{aligned} p([\] &\leftarrow \\ p([H|T]) &\leftarrow p(T) \end{aligned}$$

one can, e.g., use the information that a particular variable X is a list to abstractly unfold $p(X)$ into $p(Z) \leftarrow$, i.e. generate a single residual fact instead of the “usual” recursive definition.¹¹ This is something that *no* other specialisation framework (we are aware of) can currently achieve. In languages like Mercury [38] or Gödel [16] such type information will even be explicitly given and does not have to be inferred. We believe that our framework(s) will be especially useful for these languages.

Improved Handling of Built-in’s If we know that a given variable X represents an integer we can, e.g., specialise both *atomic*(X) or *number*(X) into *true*. One can imagine various other optimisations not possible in conventional techniques based upon the \mathcal{PD} -domain, like specialising *arg* or *functor* calls based upon type information of the arguments. A similar idea has been used in [37] and [36] to remove groundness tests (controlling parallel execution) from the residual program.

3 A Bottom-Up Analysis

Although using refined abstract domains within Algorithm 2.11 can lead to major improvements over existing specialisation techniques, it is still not possible to achieve side-ways (between different abstract conjunctions) or bottom-up success information propagation. A (seemingly) simple way to add bottom-up success information propagation to our abstract partial deduction framework is to request point 1 only for $Q \in \gamma(\mathbf{A}) \cap SS_P$ (instead of $Q \in \gamma(\mathbf{A})$) in Definition 2.1 of *aunf*(.), where SS_P is the success set of P . In practice this means that the operation *aunf*(.) can make use of a *subsidiary* (bottom-up) abstract interpretation phase to approximate SS_P . In order to achieve some interaction between the top-down and bottom-up components, one could imagine that the abstract interpretation takes the unfoldings into account (this is an approach proposed in [36]). This, however, means that one has to re-analyse whenever a new unfolding has been performed and to re-specialise whenever a tighter success set has been derived. The precise details of this “co-routining” are non-trivial and one can hardly call the above an algorithm. Furthermore, there are a considerable number of tasks (see [26]) that such an approach simply cannot handle, because the specialisation and analysis components basically still work in isolation. In this paper, we will therefore first present a pure bottom-up analysis algorithm, but which we then *fully* integrate with Algorithm 2.11 in Section 4.

¹¹A concrete specialiser might make use of non-failure analysis [8] to derive this.

In a bottom-up setting we need, instead of an abstraction of the unfolding operation, an abstraction of the bottom-up T_P operator [1, 30], or better its non-ground version (to capture the C-semantics and thus the computed answers). The (non-ground) T_P operator maps interpretations to interpretations, where an interpretation is usually represented by a set of atoms. Each interpretation in turn can be seen as representing (an approximation to) the success set. One could thus define an abstract version of T_P which maps a set of atomic abstract conjunctions to a set of atomic abstract conjunctions such that $\gamma(AT_P(\mathcal{A})) \supseteq \{H\theta_1 \dots \theta_n \mid H \leftarrow B_1, \dots, B_n \in P \wedge A_i \in \gamma(\mathcal{A}) \wedge \theta_i = \text{mgu}(B_i\theta_1, \dots, \theta_{i-1}, A_i)\}$. However, in light of a full integration with the framework of Section 2 (and in order to be able to capture all the techniques of [32]), we will describe a more refined abstraction of T_P based on *conjunctions* (instead of just atoms) and *resultants* derived by *aunf*(.) (instead of simply the clauses of the original program P).

Abstract T_P for Abstract Conjunctions

So, instead of just representing the success set for each predicate in general, we want to represent success sets for a given choice of abstract conjunctions $\mathbf{A} = \{\mathbf{A}_1, \dots, \mathbf{A}_n\}$. This is accomplished by the following definition.

Definition 3.1 An *abstracted interpretation* is a set $\{(\mathbf{A}_1, \mathbf{I}_1), \dots, (\mathbf{A}_n, \mathbf{I}_n)\}$ of couples $(\mathcal{A}Q, \mathcal{A}Q)$ ¹² such that $\mathbf{I}_i \sqsubseteq \mathbf{A}_i$ and $i \neq j \Rightarrow \mathbf{A}_i \neq \mathbf{A}_j$. We also define a projection for abstracted interpretations $\pi_1(\mathcal{I}) = \{\mathbf{A} \mid (\mathbf{A}, \mathbf{I}) \in \mathcal{I}\}$.

An abstracted interpretation \mathcal{I} represents for each \mathbf{A}_i a set of possible computed instances in the form of the abstract conjunction \mathbf{I}_i . Let us now formulate how the knowledge contained in \mathcal{I} can be used to refine some abstract conjunction, say \mathbf{A} (not necessarily identical to some \mathbf{A}_i), into another abstract conjunction $\mathbf{A}' \sqsubseteq \mathbf{A}$ approximating the success set of \mathbf{A} .

Definition 3.2 Let $\mathcal{I} = \{(\mathbf{A}_1, \mathbf{I}_1), \dots, (\mathbf{A}_n, \mathbf{I}_n)\}$ be an abstracted interpretation. Let \mathbf{A} be an abstract conjunction such that $\langle \mathbf{G}_1, \dots, \mathbf{G}_j, \dots, \mathbf{G}_m \rangle$ is an abstraction of \mathbf{A} . Also let $\mathbf{G}_j \sqsubseteq \mathbf{A}_i$ for some i . Then any abstract conjunction $\mathbf{A}' \sqsubseteq \mathbf{A}$ such that $\langle \mathbf{G}_1, \dots, \mathbf{G}_{j-1}, \mathbf{I}_i, \mathbf{G}_{j+1}, \dots, \mathbf{G}_m \rangle$ is an abstraction of it, is called a *refinement of \mathbf{A} under \mathcal{I}* . \mathbf{A} itself, as well as any refinement of \mathbf{A}' , is also called a refinement of \mathbf{A} under \mathcal{I} (i.e. we take the transitive and reflexive closure). By $\text{ref}_{\mathcal{I}}(\mathbf{A})$ we denote some refinement of \mathbf{A} under \mathcal{I} .

Example 3.3 In the \mathcal{PD} -domain let $\mathbf{A} = p(X) \wedge q(X)$ as well as $\mathcal{I} = \{(p(X), p(f(Y))), (q(f(Z)), q(f(a)))\}$. Then $\mathbf{A}' = p(f(V)) \wedge q(f(V))$ is a refinement of \mathbf{A} under \mathcal{I} . Now $\mathbf{A}'' = p(f(a)) \wedge q(f(a))$ is in turn a refinement of \mathbf{A}' (and thus also of \mathbf{A}) under \mathcal{I} . The previously inapplicable couple

¹²To extend the precision one could also use couples $(\mathcal{A}Q, 2^{\mathcal{A}Q})$, but such an extension can always be achieved by refining the abstract domain (see also [9]). Also note that every \mathbf{I}_i can be seen as an abstract substitution for \mathbf{A}_i .

$(q(f(Z)), q(f(a)))$ became applicable for \mathbf{A}' and allowed us to achieve further refinement.

We can now formulate an abstract bottom-up operator:

Definition 3.4 An *abstract bottom-up operator* $AT_P(\cdot)$ is a function $\mathcal{AQ} \times 2^{\mathcal{AQ} \times \mathcal{AQ}} \mapsto \mathcal{AQ}$ such that, for every abstracted interpretation $\mathcal{I} = \{(\mathbf{A}_1, \mathbf{I}_1), \dots, (\mathbf{A}_n, \mathbf{I}_n)\}$, we have that if $H \leftarrow B \in \text{aunf}(\mathbf{A}_i)$ and $\mathbf{B} = \text{ares}(\mathbf{A}_i, H \leftarrow B)$ then $B\theta \in \gamma(\text{ref}_{\mathcal{I}}(\mathbf{B})) \Rightarrow H\theta \in \gamma(AT_P(\mathbf{A}_i, \mathcal{I}))$.

Intuitively, the above states that if a runtime resolvent ($\leftarrow B\theta$) of \mathbf{A}_i may succeed given the abstracted interpretation \mathcal{I} (i.e., $B\theta \in \gamma(\text{ref}_{\mathcal{I}}(\mathbf{B}))$) then the corresponding head $H\theta$ should potentially succeed in $AT_P(\mathbf{A}_i, \mathcal{I})$. In other words, $AT_P(\mathbf{A}_i, \mathcal{I})$ is a safe approximation of one concrete non-ground¹³ bottom-up propagation step performed on the resultants of \mathbf{A}_i . We also define $AT_P(\cdot)$ to work on abstracted interpretations: $AT_P(\mathcal{I}) = \{(\mathbf{A}_1, AT_P(\mathbf{A}_1, \mathcal{I})), \dots, (\mathbf{A}_n, AT_P(\mathbf{A}_n, \mathcal{I}))\}$.

For an abstract domain with no infinite ascending chains, we can now formulate a terminating bottom-up analysis algorithm basically as calculating $AT_P \uparrow^\infty (\mathcal{I}_0)$ where $\mathcal{I}_0 = \{(\mathbf{A}_1, \perp), \dots, (\mathbf{A}_n, \perp)\}$.

Example 3.5 Take the program $P = \{C_1, C_2, C_3\}$ from Example 2.4 and let $\mathcal{I}_0 = \{(\mathbf{A}, \perp)\}$ with $\mathbf{A} = p(\text{any})$ and using an abstract domain with type information. Also let $\text{aunf}(\mathbf{A}) = \{C_1, C_2, C_3\}$, $\text{ares}(A, C_1) = \square$ and $\text{ares}(\mathbf{A}, C_2) = \text{ares}(\mathbf{A}, C_3) = \mathbf{A}$. To calculate $I_{j+1} AT_P(\mathbf{A}, \mathcal{I}_j)$ we then obtain:

- $AT_P(\mathbf{A}, \mathcal{I}_0) = \mathcal{I}_1 = \{(\mathbf{A}, p(\tau = a))\}$ is correct: $p(a) \in \gamma(AT_P(\mathbf{A}, \mathcal{I}_0))$ holds and for C_2 and C_3 we have $\text{ref}_{\mathcal{I}_0}(\mathbf{A}) = \perp$. Observe that $\mathcal{I}_1 = \{(\mathbf{A}, p(\text{any}))\}$ would also conform to Definition 3.4 (but is obviously much less precise).
- $\mathcal{I}_2 = \mathcal{I}_3 = \{(\mathbf{A}, p(\tau = a \mid f(\tau) \mid g(\tau)))\}$ is admissible. For C_2 and C_3 we now have $\text{ref}_{\mathcal{I}_1}(\mathbf{A}) = p(\tau = a)$ and $\text{ref}_{\mathcal{I}_2}(\mathbf{A}) = p(\tau = a \mid f(\tau) \mid g(\tau))$ respectively. Hence we must have $p(f(a)) \in \gamma(AT_P(\mathbf{A}, \mathcal{I}_1))$ as well as $p(g(a)) \in \gamma(AT_P(\mathbf{A}, \mathcal{I}_1))$. At the next iteration we then must have $p(f(\bar{t})) \in \gamma(AT_P(\mathbf{A}, \mathcal{I}_2))$ and $p(g(\bar{t})) \in \gamma(AT_P(\mathbf{A}, \mathcal{I}_2))$ where \bar{t} is any term of type $\tau = a \mid f(\tau) \mid g(\tau)$. All of the above hold. Observe that $\mathcal{I}_2 = \{(\mathbf{A}, p(\tau = a \mid f(a) \mid g(a)))\}$ is also admissible (but then no fixpoint is reached).

Exploiting Success Information for the Code Generation

The following shows how the information derivation by such an analysis can be exploited to derive a specialised program.

¹³ $H\theta$ and $B\theta$ are not necessarily ground.

Definition 3.6 Let \mathcal{I} be an abstracted interpretation. We then define an *abstract partial deduction of P wrt \mathcal{I}* to be the set of clauses:

$$\{\rho_{\mathbf{A}}(H\theta) \leftarrow \rho_{\mathcal{A}, \mathbf{A}''}(B\theta) \mid H \leftarrow B \in \text{aunf}(\mathbf{A}) \wedge \mathbf{A} \in \pi_1(\mathcal{I}) \wedge \mathbf{A}'' = \text{ref}_{\mathcal{I}}(\text{ares}(\mathbf{A}, H \leftarrow B)) \wedge B\theta \text{ is a concrete dominator}^{14} \text{ of } \mathbf{A}''\}$$

If for all \mathbf{A}'' we have that $\{\mathbf{A}''\} \sqsubseteq_{\text{split}} \pi_1(\mathcal{I})$ then we call \mathcal{I} *covered* (and all renamings $\rho_{\mathcal{A}, \mathbf{A}''}(B\theta)$ above are defined).

The big difference over Definition 2.9 is that the resultants get instantiated using the success information contained in \mathcal{I} and that the notion of coveredness also takes the success information into account. Indeed, \mathcal{I} might be covered even though $\pi_1(\mathcal{I})$ is not (we might have $\{\mathbf{A}'\} \not\sqsubseteq_{\text{split}} \pi_1(\mathcal{I})$).

Example 3.7 Let P be the program from Example 2.2 and $\mathcal{I} = \{(eq(X, Y), eq(X, X))\}$ in the \mathcal{PD} -domain. Also, let $\text{aunf}(eq(X, Y)) = \{C_1, C_2\}$ and $\text{ares}(eq(X, Y), C_2) = eq(X, Y)$ with $C_1 = eq([], []) \leftarrow$ as well as $C_2 = eq([H|S], [H|T]) \leftarrow eq(S, T)$. Then, obviously, $eq(X, X)$ is a refinement of $eq(X, Y)$ and the following is an abstract partial deduction of P wrt \mathcal{I} (using $\rho_{eq(X, Y)} = eq_1(X, Y)$):

$$\begin{aligned} eq_1([], []) &\leftarrow \\ eq_1([H|T], [H|T]) &\leftarrow eq_1(T, T) \end{aligned}$$

The main technique of [32] can be seen an instance of this analysis by: taking the \mathcal{PD} -domain and using a composition of predicate-wise *msg* with the non-ground T_P operator on conjunctions. However, only a simple one-step unfolding is performed in [32] and it is not allowed to further refine refinements (which can be crucial, see [26]).

The calculation of the least fixpoint of non-ground T_P can of course also be seen as an instance of this approach (in that case $\mathcal{AQ} = 2^{\mathcal{Q}}$ and the \mathbf{A}_i of \mathcal{I}_0 are initialised by singleton sets of maximally general atoms, $\text{aunf}(\cdot)$ performs a single unfolding step and $AT_P(\mathbf{A}_i, \mathcal{I}) = \{H\theta_1 \dots \theta_n \mid H \leftarrow B_1, \dots, B_n \in P \wedge \theta_i = \text{mgu}^*(B_i\theta_1 \dots \theta_{i-1}, A_i)$ with $B_i \preceq \mathbf{A}_j$ and $A_i \in \mathbf{I}_j\}$.

Theorem 3.8 Let P' be an abstract partial deduction of P wrt $AT_P \uparrow^\infty(\mathcal{I})$. Let $Q \in \gamma(\mathbf{A})$, $\mathbf{A} \in \pi_1(\mathcal{I})$, $AT_P \uparrow^\infty(\mathcal{I})$ be covered and $\text{aunf}(\cdot)$ be conservative.¹⁵ Then $P \cup \{\leftarrow Q\}$ has an SLD-refutation with c.a.s. θ iff $P' \cup \{\leftarrow \rho_{\mathbf{A}}(Q)\}$ has.

¹⁴One could also allow a set of instantiations $\theta_1, \dots, \theta_n$ such that all concretisations of \mathbf{A}'' are an instance of at least one atom in $\{B\theta_1, \dots, B\theta_n\}$. This can lead to more instantiated resultants but might also lead to code duplication and considerable slow-downs.

¹⁵The difference with Theorem 2.10 is that calls to predicates are no longer guaranteed to be concretisations of the abstract conjunctions from which their definition has been derived; only their success patterns are! Therefore the code also has to be sound (but not complete) for calls which are not concretisations. An alternative is to allow non-conservative rules but use the refinements only in, e.g., a left-to-right fashion.

Proof In Appendix B. □

For finite failure we can also derive that, if $ainf(\cdot)$ is fair then if $P \cup \{\leftarrow Q\}$ has a finitely failed SLD-refutation then so does $P' \cup \{\leftarrow \rho_{\mathbf{A}}(Q)\}$ (but not necessarily the other way around).

One major problem is now of course how to find interesting sets of abstract conjunctions $\mathcal{A} = \{\mathbf{A}_1, \dots, \mathbf{A}_n\}$ (this was left open in [32]) as well as how to ensure that $AT_P \uparrow^\infty (\{(\mathbf{A}_1, \perp), \dots, (\mathbf{A}_n, \perp)\})$ is covered. Here the top-down framework can help, which in turn can benefit from the information provided by the bottom-up phase. This full integration is developed in the next section.

4 A Combined Top-Down/Bottom-Up Framework

The idea of the following algorithm is to combine the top-down with the bottom-up approach so that the mutually benefit from each other:

- the top-down component can, in addition to propagating goal dependent information downwards, provide interesting sets of abstract conjunctions $\{\mathbf{A}_1, \dots, \mathbf{A}_n\}$ for the bottom-up phase and ensure coveredness.
- the bottom-up phase can give the top-down component information about the global success-patterns, allowing a more focussed unfolding, producing more instantiated resultants as well as achieving side-ways information passing.

As shown in [26], for a particular abstract domain, such an integration can achieve optimisation and analysis which cannot be derived by either approach alone, nor by combining them in a naive manner (i.e. running them successively in isolation, as, e.g., discussed at the beginning of Section 3).

To formalise the flow analysis component of our integrated algorithm we define a refined abstract unfolding and resolution operator, which takes the current success information into account: $aiunf^*(\mathcal{I}) = \{(\mathbf{L}', \perp) \mid (\mathbf{A}, \mathbf{I}) \in \mathcal{I} \wedge H \leftarrow B \in ainf(\mathbf{A}) \wedge \mathbf{L}' = ref_{\mathcal{I}}(ares(\mathbf{A}, H \leftarrow B))\}$. We also extend ω to abstracted interpretations and request that $\pi_1(\omega(\mathcal{I})) \sqsubseteq_{split} \pi_1(\mathcal{I})$.

Algorithm 4.1 (Refined Abstract Partial Deduction)

Input: A program P and an abstract conjunction \mathbf{A}

Output: A specialised program P'

Initialise: $i = 0, \mathcal{I}_0 = \{(\mathbf{A}, \perp)\}$

repeat

let $j := i; \mathcal{I}_{i+1} := AT_P(\mathcal{I}_i);$ **let** $i := i + 1;$ /* one BUP step */

repeat

let $\mathcal{I}_{i+1} := \omega(\mathcal{I}_i \cup aiunf^*(\mathcal{I}_i));$ **let** $i := i + 1;$

until $\mathcal{I}_{i-1} = \mathcal{I}_i$

until $\mathcal{I}_j = \mathcal{I}_i$

Let P' be an abstract partial deduction wrt P and \mathcal{I}_i

One can easily see that, once the algorithm has terminated, \mathcal{I}_i is covered. In fact, the inner repeat-loop — performing top-down abstract partial deduction — ensures (refined) coveredness. Also, abstract unfolding is applied after every single bottom-up step, i.e. *before* the fixpoint of $ATP(\cdot)$ is reached. This is the important aspect which makes this algorithm more powerful than running the top-down and bottom-up components in isolation (see [22] for a fully worked out example in the \mathcal{PD} -domain).

Example 4.2 Let P be the following program:

```

(C1) p(a) ←
(C2) p(f(X)) ← p(X)
(C3) p(g(X)) ← p(X)
(C4) q(a) ←
(C5) q(g(X)) ← q(X)
(C6) t(X) ← q(X), p(X)

```

A trace of Algorithm 4.1 is now as follows:

- $\mathcal{I}_0 = \{(t(\text{any}), \perp)\}$
- $\mathcal{I}_1 = \{(t(\text{any}), \perp), (q(\text{any}), \perp), (p(\text{any}), \perp)\}$
- $\mathcal{I}_2 = \{(t(\text{any}), \perp), (q(\text{any}), q(a)), (p(\text{any}), p(a))\}$
- $\mathcal{I}_3 = \{(t(\text{any}), t(a)), (q(\text{any}), q(\tau = a \mid g(\tau))),$
 $(p(\text{any}), p(\tau = a \mid f(\tau) \mid g(\tau)))\}$
- $\mathcal{I}_4 = \{(t(\text{any}), t(\tau = a \mid g(\tau))), (q(\text{any}), q(\tau = a \mid g(\tau))),$
 $(p(\text{any}), p(\tau = a \mid f(\tau) \mid g(\tau)))\}$
- $\mathcal{I}_5 = \mathcal{I}_4 \cup \{(p(\tau = a \mid g(\tau)), \perp)\}$

The algorithm in [26, 22] is an instance of the above algorithm using the \mathcal{PD} -domain and where $ATP(\cdot)$ is the predicate-wise *msg* composed with the non-ground TP operator. Algorithm 4.1 is also strictly more powerful than [14, 7] (which uses the analysis information just to remove redundant clauses, and not, e.g., to instantiate them; it can actually be seen as running classical partial deduction first and then some analysis afterwards) or [36] (which cannot perform deforestation or tupling as it is restricted to specialising atoms individually). One can actually also express techniques based upon tabling (OLDT [40, 19] or even EOLDT [2]) in a slight extension of our framework. One simply has to allow a single abstract conjunction \mathbf{A} to be also abstracted by a set $\{\mathbf{A}_1, \dots, \mathbf{A}_n\}$ of abstract conjunctions covering all its concretisations (i.e. $\gamma(\mathbf{A}) \subseteq \bigcup_{1 \leq i \leq n} \gamma(\mathbf{A}_i)$). This is required as tabling will generate new entries for every distinct call. Then one can use the powerset of the concrete domain (all \mathbf{A}_i within abstracted interpretations will then actually be singleton sets representing the calls and each \mathbf{I}_i represents the table of answers for \mathbf{A}_i). We then have:

- $\mathcal{AQ} = 2^{\mathcal{Q}}$, i.e. the abstract domain is the power set of the concrete domain, However, all \mathbf{A}_i within abstracted interpretations will actually be singleton sets (but the \mathbf{I}_i can be sets, describing the computed

- answers, i.e. the table for \mathbf{A}_i) and abstract elements represent all their instances,
- the initial \mathbf{A} is a singleton set $\{A\}$, where A is an atom, and ω will systematically split conjunctions into singleton sets of atoms (and in case term-depth abstraction is used, will also perform the proper structure generalisation),
- abstract unfolding is the same as a 1-step concrete unfolding,
- $AT_P(\cdot)$ is the non-ground T_P operator (in case answer abstraction is performed some generalisation can appear here as well).

In other words, the reconciliation of bottom-up and top-down evaluation [4] is just a special case of our reconciliation of specialisation and analysis. To express EOLDT [2] we simply drop the requirement that ω only produces atomic conjunctions and that the initial $\{A\}$ is an atom.

A Worked-Out Example

Let P be the following program (from an open problem in [26, 22]):

```

rev_last(L, X) ← rev(L, [a], R), last(R, X)
rev([], L, L) ←
rev([H|T], Acc, Res) ← rev(T, [H|Acc], Res)
last([X], X) ←
last([H|T], X) ← last(T, X)

```

This example encapsulates the essence of problems that arise when statically known values (a) are stored in a dynamic data-structure (L). In practice, this data-structure can, e.g., be the environment used by an interpreter or the substitution in an explicit unification algorithm. Being able to “retrieve” the static value (a) is vital if any serious specialisation is to take place. Unfortunately no existing analysis, specialisation or transformation technique we are aware of, is able to solve this problem.

A similar problem, the *append-last* problem $append(L, [a], R), last(R, X)$, has been successfully tackled in [26, 22] by combining conjunctive partial deduction with bottom-up abstract interpretation in the \mathcal{PD} -domain. The crucial ingredient of success lay in the fact that conjunctive partial deduction was able to deforest (i.e. remove) the intermediate list R (whose structure was too complex for the abstract domain under consideration). This in turn allowed the bottom-up component to infer that X is a (in all successful derivations).

However, in the above program rev is written using an accumulating parameter, and in that case neither conjunctive partial deduction, nor any unfold/fold method we know of, can deforest the intermediate variable R (see [22]) and no existing technique is able to derive that, in all answers to $rev_last(L, X)$, X will be bound to a . We will show how this problem can be solved in our framework in a rather straightforward manner.

When unfolding $rev(L, [a], R), last(R, X)$ one will encounter the conjunction $rev(L', [H, a], R), last(R, X)$. If we continue to unfold rev then the ac-

cumulator will simply continue to grow. As R does not get instantiated, unfolding $last$ is not of much help either. So, all one can do to ensure termination, is to abstract the accumulator. Unfortunately, in the \mathcal{PD} -domain, the msg of $[a]$ and $[H, a]$ is $[H|T]$ and the information that a is the last element of the list has been lost.

In our framework, however, we can in addition to specialising conjunctions, provide much more refined generalisation. As already hinted at in Section 2.7, we can, e.g., use the homeomorphic embedding relation [39, 28, 15] in a straightforward manner to produce type information describing the growth it detected. More precisely, in our case we have $[a] \trianglelefteq [H, a]$ as we can “strike out” the $[H|\cdot]$ in $[H, a]$ in order to obtain $[a]$ (and \trianglelefteq thus tells us that there is a growth of structure and we should generalise). Now, all we have to do is to use the information provided by \trianglelefteq and extrapolate the growth. This leads to the generalisation $A : \tau$ where the type τ is defined as $\tau ::= [a] \mid [any]\tau$.

If we now abstractly unfold $rev(L, A : \tau, R) \wedge last(R, X)$ we might get, after also specialising $last(A : \tau, X)$ and renaming:

$$\begin{aligned} rev_last(L, X) &\leftarrow rl(L, [a], R, X) \\ rl([], A, A, X) &\leftarrow last(A, X) \\ rl([H|T], A, R, X) &\leftarrow rl(T, [H, A], R, X) \\ last([a], a) &\leftarrow \\ last([H|T], X) &\leftarrow last(T, X) \end{aligned}$$

Even if we just use non-ground T_P for $AT_P(\cdot)$, we will get as our final abstract partial deduction:

$$\begin{aligned} rev_last(L, a) &\leftarrow rl(L, [a], R, a) \\ rl([], A, A, a) &\leftarrow last(A, a) \\ rl([H|T], A, R, a) &\leftarrow rl(T, [H, A], R, a) \\ last([a], a) &\leftarrow \\ last([H|T], a) &\leftarrow last(T, a) \end{aligned}$$

In other words, we have succeeded in deriving the desired information. It is of course possible to further optimise this program. E.g., in case we additionally know that L is a list, we can actually use our extended abstract unfolding possibilities (c.f. Example 2.3) together with the post-processing of [29] to generate the following optimal residual code:

$$rev_last(L, a) \leftarrow$$

It is also possible to use the same approach to prove inductive theorems in a much less ad-hoc (and more generally reusable manner) than, e.g., [41]. We also believe that automation of this approach is feasible and endeavour to incorporate these possibilities into the ECCE partial deduction system in the not too distant future. We believe that the improved specialisation capabilities conferred by our new framework will further extend the practical applicability of program specialisation (especially since a lot of practical programs use accumulating parameters).

5 Future Work and Conclusion

A lot of avenues can be pinpointed for further work. First, on the practical side, one should of course fully work out and implement useful instances of the generic algorithms presented in this paper. Domains based upon types (or type graphs), inferring these from the homeomorphic embedding, look very promising. On the theoretical side, one can try to handle logic programs with negation. Observe, however, that the algorithm of Section 4 can replace infinite failure by finite failure. One should therefore concentrate on the well-founded semantics and SLS [35] and not on the completion semantics or SLDNF. One can also endeavour to add ever more powerful, but ever more difficult to automate, methods such as goal replacement, specialising disjunctions of conjunctions [34] or specialising conjunctions of unlimited length [33].

In this paper we have presented a generic framework and algorithm for top-down program specialisation, which supersedes earlier top-down approaches in generality and power. We have established a *generic correctness result* and have shown how the additional power can be exploited in practice, for improved generalisation, unfolding and code-generation. We have also clarified the relationship of top-down partial deduction with abstract interpretation, establishing a *common basis* and terminology. This clarification allowed us to precisely pinpoint shortcomings both of existing top-down specialisation methods and of existing abstract interpretation techniques. We then proceeded to remedy these shortcomings by incorporating bottom-up success information propagation, thereby fully *reconciling program specialisation with abstract interpretation* and providing a *unifying framework* into which almost all existing specialisation techniques can be cast. This new integrated framework with its generic algorithm provides the foundation for new, powerful specialisation *and* analysis outside the scope of existing techniques.

Acknowledgements

Maurice Bruynooghe provided very valuable feedback this paper. The author also greatly benefited from discussions with Danny De Schreye, Neil Jones, Jesper Jørgensen, Bern Martens, Torben Mogensen, Morten Heine Sørensen, and comments of anonymous referees.

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A Proof of Theorem 2.10

Proof (Sketch) Both the proof of soundness and completeness are by induction on the length of the refutations. For the induction to go through we need to slightly generalise the theorem into the induction hypothesis:

Let P' be an abstract partial deduction wrt a covered set of abstract conjunctions \mathcal{A} and let $Q_i \in \gamma(\mathbf{A}_i)$ with $\mathbf{A}_i \in \mathcal{A}$. Then

- $P' \cup \{\leftarrow Q_1 \wedge \dots \wedge Q_n\}$ has an SLD-refutation with c.a.s. θ iff $P' \cup \{\leftarrow \rho_{\mathbf{A}_1}(Q_1) \wedge \dots \wedge \rho_{\mathbf{A}_n}(Q_n)\}$ has an SLD-refutation with c.a.s. θ .

Also, to simplify the presentation of the proof we suppose that for all renamings we have $\rho_{\mathbf{A}}(Q)\theta = \rho_{\mathbf{A}}(Q\theta)$ and that, if $\rho_{\mathcal{A},\mathbf{A}}(B)$ is defined, $\rho_{\mathcal{A},\mathbf{A}}(B)\theta = \rho_{\mathcal{A},\mathbf{A}}(B\theta)$. If these properties are not verified then we have

to proceed like in [25] and introduce the concept of “admissible renamings” and several renaming functions.

← (**soundness of P'**): We proceed by induction on the length of the refutation δ for $P' \cup \{\leftarrow \rho_{\mathbf{A}_1}(Q_1) \wedge \dots \rho_{\mathbf{A}_n}(Q_n)\}$. The base case ($len = 0$ and thus $n = 0$) is trivial. For the induction step let us examine the first resolution step of δ resolving an atom $\rho_{\mathbf{A}_i}(Q_i)$ with a clause $\rho_{\mathbf{A}_i}(H) \leftarrow \rho_{\mathbf{A},\mathbf{B}}(B)$ via *mgu* θ_1 . We know, by properties of the renaming that θ_1 is also an *mgu* of Q_i and H . The if-part (\Leftarrow) of point 1 of Definition 2.1 (defining *aunf*(.)) therefore ensures that we can find a (possibly incomplete) SLD-derivation for $P \cup \{\leftarrow Q_i\}$ leading to the goal $\leftarrow B\theta_1$ via computed answer θ_1 . By point 2.2 of the same definition (defining *ares*(.)) we know that $B\theta_1 \in \gamma(\mathbf{B})$, as $B = ares(\mathbf{A}_i, H \leftarrow B)$ and $Q_i \in \gamma(\mathbf{A}_i)$. Thus, applying the sub-derivation lemma of [31], we know that the resolvent in P' is:

$$\leftarrow \rho_{\mathbf{A}_1}(Q_1)\theta_1 \wedge \dots \rho_{\mathbf{A},\mathbf{B}}(B)\theta_1 \wedge \dots \rho_{\mathbf{A}_n}(Q_n)\theta_1$$

while the resolvent in P is:

$$\leftarrow Q_1\theta_1 \wedge \dots B\theta_1 \wedge \dots Q_n\theta_1$$

By our assumption about renamings, we know that $\rho_{\mathbf{A}_n}(Q_n)\theta_1 = \rho_{\mathbf{A}_n}(Q_n\theta_1)$ as well as $\rho_{\mathbf{A},\mathbf{B}}(B)\theta_1 = \rho_{\mathbf{A},\mathbf{B}}(B\theta_1)$. We also know, as all our abstract conjunctions are downwards closed, that $Q_i\theta$ is still a concretisation of \mathbf{A}_i . We can thus apply the induction hypothesis for the resolvent goals and have thus established soundness.

⇒ (**completeness of P'**): We now proceed by induction on the length of the refutation δ for $P \cup \{\leftarrow Q_1 \wedge \dots Q_n\}$. The base case ($len = 0$ and thus $n = 0$) is again trivial. For the induction step, let us choose some Q_i . As $Q_i \in \gamma(\mathbf{A}_i)$ we can apply Definition 2.1 of *aunf*(.) to deduce that there is an SLD-tree τ for $P \cup \{\leftarrow Q_i\}$ such that points 1 and 2 hold. By independence of the selection rule ([1, 30]) we know that we do not lose any computed answers by enforcing a particular selection rule. We will thus only consider those derivations δ which unfold some $\leftarrow Q_i$ in the manner prescribed by τ .¹⁶ Now, the only-if part (\Leftarrow) of point 1 of Definition 2.1, together with the fact that renaming preserves the *mgu*, states that any (partial) computed answer θ_1 (leading to the resolvent $\leftarrow B$) obtained for $\leftarrow Q_i$ in τ can also be obtained by resolving $\rho_{\mathbf{A}_i}(Q_i)$ with a clause $\rho_{\mathbf{A}_i}(H) \leftarrow \rho_{\mathbf{A},\mathbf{B}}(B)$ in P' . Thus, applying the sub-derivation lemma of [31], we know that the resolvent in P is:

$$\leftarrow Q_1\theta_1 \wedge \dots B\theta_1 \wedge \dots Q_n\theta_1$$

while the resolvent in P' is:

$$\leftarrow \rho_{\mathbf{A}_1}(Q_1)\theta_1 \wedge \dots \rho_{\mathbf{A},\mathbf{B}}(B)\theta_1 \wedge \dots \rho_{\mathbf{A}_n}(Q_n)\theta_1$$

¹⁶If we want to establish the preservation of finite failure it is vital that the unfoldings performed by τ are fair. For computed answers, however, this does not matter.

By the if-part of point 1 of Definition 2.1 we know that $B\theta_1 \in \gamma(\mathbf{B})$ and as all our abstract conjunctions are downwards closed, we also know that $Q_i\theta$ is still a concretisation of \mathbf{A}_i . Also, by our assumption about renamings, we know that $\rho_{\mathbf{A}_n}(Q_n)\theta_1 = \rho_{\mathbf{A}_n}(Q_n\theta_1)$ as well as $\rho_{\mathcal{A},\mathbf{B}}(B)\theta_1 = \rho_{\mathcal{A},\mathbf{B}}(B\theta_1)$. Therefore, we can apply the induction hypothesis on the resolvent goals in order to establish completeness. \square

B Proof of Theorem 3.8

Proof (Sketch) The difference over Theorem 2.10 is that for every $H \leftarrow B \in \text{aunf}(\mathbf{A})$ and $\mathbf{A}' \in \text{ares}(\mathbf{A}, H \leftarrow B)$, where $(\mathbf{A}, \mathbf{I}) \in \mathcal{I}$, only a refinement $\mathbf{A}'' = \text{ref}_{\mathcal{I}}(\mathbf{A}')$ of \mathbf{A} is guaranteed to be covered by \mathcal{I} . I.e. a runtime call to a residual predicate generated for \mathbf{A}'' (or an abstraction thereof) is no longer guaranteed to be a concretisation of \mathbf{A}'' . Thus, to establish that this does not destroy correctness proved by Theorem 2.10 we have to:

1. establish that code generated for \mathbf{A}'' is sound (i.e. no extra answers are produced) also for calls which are not concretisations of \mathbf{A}'' . This is a direct consequence of our requirement that $\text{aunf}(\cdot)$ is conservative,
2. establish that \mathbf{I} is a safe approximation of the success set for the concretisations of \mathbf{A} . This is a consequence of the safety of $\text{aunf}(\cdot)$ requested in Definition 3.4.
3. establish that computed answers are not affected by applying the substitution θ in Definition 3.6 leading to the resultant $\rho_{\mathbf{A}}(H\theta) \leftarrow \rho_{\mathcal{A},\mathbf{A}''}(B\theta)$. This follows from the fact that, by point 2, all computed instances of B are instances of $B\theta$. $\rho_{\mathbf{A}}(H\theta) \leftarrow \rho_{\mathcal{A},\mathbf{A}''}(B\theta)$ is thus a more specific version of $\rho_{\mathbf{A}}(H) \leftarrow \rho_{\mathcal{A},\mathbf{A}''}(B)$ in the sense of [32] and thus, by the results of [32], the set of computed answers is preserved (as is finite failure, but not infinite one).

\square